Some Studies on Lie Algebra Bundles

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1 Introduction

The theory of Lie algebras has seen new impetuous developments in recent years. From physics, to differential geometry and algebraic topology these structures are proved to be interesting and produced innovative ideas and methods that have helped to solve critical problems. Lie algebra bundles are essentially fibre bundles whose fibres are finite dimensional Lie algebras and local triviality respects the algebra structure.

2 Objectives

In this thesis we have planned the following major objectives.

1. To establish the theory of algebraic Lie algebra bundles.
2. To characterize the derivations, in particular, inner and central derivations of a Lie algebra bundle and study the structural relations between a Lie algebra bundle and its derivation algebra bundle.
3. To define prime, semiprime, irreducible and primary ideal bundles of a Lie algebra bundle and study their interrelations.
4. To obtain an elemental characterization of strong primeness in Lie algebra bundles.

3 Outline of the thesis

The thesis "Some Studies on Lie Algebra Bundles" is divided into seven chapters.

Chapter 1 : INTRODUCTION

Chapter 2 : ALGEBRAIC LIE ALGEBRA BUNDLES

Chapter 3 : DERIVATIONS OF LIE ALGEBRA BUNDLES

Chapter 4 : PRIME, SEMIPRIME AND IRREDUCIBLE IDEAL BUNDLES

Chapter 5 : PRIMARY IDEAL BUNDLES

Chapter 6 : STRONG PRIMENESS IN LIE ALGEBRA BUNDLES

Chapter 7 : CONCLUSION AND SCOPE FOR FUTURE WORK

3.1 CHAPTER 1: INTRODUCTION

Vector bundles are a mathematical generalization of the Cartesian product.The concept of fiber bundles became an object of research as a result of the many issues addressed in

1930s about the geometry and topology of manifolds. Some attribute the emergence of vector bundles to the paradigm change from Aristotelian physics to Galilean physics. As Dale Husemoller in [39] states, beginning with Hassler Whitney's initial general definition, the notion of fiber bundle had been properly stated by 1950, and the homotopy classification of fiber bundles had been achieved. Following the works of A. Grothendieck, Atiyah and Hirzebruch developed the branch of K-theory in the early 1960s.

For a vector bundle ξ sections of ξ give rise to a generalization of vector valued functions defined on the topological space X. So in particular, if we consider the tangent bundle of a vector bundle, the sections will be all vector fields. In the field of algebraic geometry we can find only a few globally defined functions so this general notion aid in algebraic geometry. Vector bundles is an integral part of differential geometry and also of great importance in modern physics such as in gauge theory and Yang Mill's theory [7][87].

Vector bundles originated in mathematics before quantum theory arose in physics [96]. For instance, geometric quantization [64] is a mathematical method for defining a quantum theory that corresponds to a given classical theory in mathematical physics. In this theory, one usually considers vector bundles over a symplectic manifold. The symplectic manifold represents the state space of a classical mechanical system, for example in the case of a particle moving on a line, the symplectic manifold is R^2, the space of the particle's positions and momenta. Since they are derived from Poisson brackets, the equations of motion in classical mechanics are in general nonlinear in state space coordinates. The quantum version of this problem takes into account line bundles in classical symplectic space. The wave functions of the particle are represented by the sections of these bundles. The equations of motion in quantum mechanics are linear. This linearization is accomplished by working on the line bundle, whose intrinsic structure is linear, and the induced evolution operators acting linearly on the section space.

Moduli spaces of flat connections define quantizable manifolds with very interesting quantum theory. These flat connections are the classical solutions of Chern-Simons theories. Sections of spinor bundles describe fermions on curved spaces. Berry phases describe holonomies of fiber bundle connections. Sections of vector bundles describe Higgs fields. Richard Montgomery's "The gauge theory of the falling cat"[78] discusses the need for fibre bundles in the description of classical flexible systems.

In the last century, research on deformation theory [26, 27, 28, 29] has become increasingly significant in addition to studies on fiber bundles and sheaves. On the other hand, Lie algebras connect most branches of mathematics and physics.

The solution of differential equations also motivates the study of Lie algebras. Similar to Galois' work on polynomial equations, understanding these symmetries might aid in comprehending the solutions of these equations. There are several applications for Lie algebras defined over fields of prime characteristic. They are very useful in algebraic geometry, for example, in affine group schemes, group theory, solving the restricted Burnside problem and purely inseparable fields extensions. Since the last part of the 20th century, an algebraic approach based on the idea of dynamical symmetry, through the language of Lie algebras has been used in molecular spectroscopy.

According to Elie Cartan's third theorem of Lie,"every Lie algebra of finite dimension is isomorphic to the Lie algebra of a Lie group." J. P. Serre, an eminent mathematician

who was awarded mathematics' highest honor, the Fields Medal, in 1954 and the first recipient of the Abel Prize for lifetime achievement, posed the question of whether the third theorem of Lie was true for Lie algebras and Lie groups that depend continuously on a parameter. In 1966, Douady and Lazard answered this question by introducing the concept of Lie algebra and Lie group bundles[22]. Beginning with this, Lie algebra and Lie algebra bundles have arisen as topics of increased interest.

Working on the problem left open by Douady and Lazard, a study of real (complex) Lie algebra bundle was carried out by Kiranagi [50]. In [50] he demonstrated that if every fibre in a Lie algebra bundle is mutually isomorphic over a topological space then the bundle is locally trivial. This theorem helped to extend most of the properties of vector bundles to the case of Lie algebra bundles. Further the converse also was shown to hold good. He has also proved that in the case when the underlying field has characteristic zero, a semisimple Lie algebra bundle will be the direct sum of simple Lie algebra bundles. [51]. G. Prema and B. S. Kiranagi demonstrated the definitions of semisimple Lie algebra bundles and of module bundles, as well as the corresponding decomposition theorems. [56]. Furthermore, they have shown that a locally trivial Lie algebra bundle is a semidirect product of the radical bundle and a semisimple subalgebra bundle by employing cohomological techniques. [57]. B. S. Kiranagi, Ranjitha Kumar, and G. Prema [62] have enhanced the theory of Lie algebra bundles over any field in terms of characteristic ideal bundles, following Seligman [85] and Lu Chaihui[10]. The relationship between homotopy theory and the pullback, clutching, and collapsing constructions of vector bundles[6] to Lie algebra bundles has been examined by R.Rajendra, Ranjitha Kumar and B.S. Kiranagi [83]. Béchir Dali [19] constructed several examples of Lie algebra bundles on the space X, where X is contained in $M_{m\times n}R$. Many authors along with Kiranagi worked on Lie, associative and Jordan algebra bundles [50, 51, 56, 57, 58, 53, 52, 60, 62, 55, 59, 72, 82].

Studies on derivations of algebra bundles was made by B. S. Kiranagi, Ranjitha Kumar, K. Ajaykumar and B. Madhu in [63].

We observe that the study of vector bundles has emerged as a subject of larger study. Considering the fibres as Lie algebras, one can analyse the vector bundles in algebraic as well as topological point of view.

3.1.1 Outline of the thesis

The content of the thesis is divided into seven chapters including the introductory chapter.

Chapter 1: It is an introductory chapter that contains history and motivation for the study. It is intended to introduce basic definitions, terminologies and concepts related to our study.

Chapter 2: The algebraicity of Lie algebra bundles and their derivation algebra bundles are investigated in this chapter.

Chapter 3: This chapter discusses characteristically solvable Lie algebra bundles and examines the properties and structural relations of inner and central derivations with the Lie algebra bundle.

Chapter 4: This chapter studies the relationships among the prime, semiprime, irreducible, and maximal ideal bundles of a Lie algebra bundle. Characterization of prime ideal bundles in terms of sections of Lie algebra bundles is obtained.

Chapter 5: Primary ideal bundles of a Lie algebra bundle are studied and its relation with prime, semiprime and irreducible ideal bundles are discussed here. Characterization of primary ideal bundles in terms of sections of Lie algebra bundles is achieved. Also, the concepts of prime Lie algebra bundles and primary Lie algebra bundles are introduced briefly.

Chapter 6: In this chapter, strong primeness in Lie algebra bundles is discussed and an elemental characterization of strong primeness is deduced.

3.1.2 Basic definitions and preliminaries

This section contains the basic definitions, notations and ideas that will be utilized throughout the thesis and is very essential for the development of the thesis.

Definition 3.1. *An Algebra over a field F is a vector space over F along with a bilinear multiplication $(x, y) \mapsto xy$. A Lie algebra is an algebra L in which for all x in L, $x^2 = 0$ and for all x, y, z in L, the Jacobi identity $(xy)z + (yz)x + (zx)y = 0$ is satisfied . The Lie multiplication is denoted by $[\cdot, \cdot]$.*

Definition 3.2. *A Lie algebra L is abelian if for all x, y in L, the Lie multiplication, $[x, y] = 0$.*

Definition 3.3. *A subspace of an algebra is a subalgebra if the subspace is closed under the multiplication operation of the algebra.*

Definition 3.4. *A subspace I of an algebra A, is an ideal if for all $x \in I$ and $y \in A$, $xy \in I$.*

Definition 3.5. *An algebra A taken over a field F is said to be simple if A has $\{0\}$ and A as the only ideals of and A is not the zero algebra (this condition is imposed in order to avoid giving undue prominence to the one dimensional algebra).*

Definition 3.6. *If A is any algebra over a field F we obtain a derived series of subalgebras as $A^{(1)} \supseteq A^{(2)} \supseteq A^{(3)} \supseteq A^{(4)} \dots$ by defining $A^{(1)} = A$ and $A^{(k+1)} = \left(A^{(k)}\right)^2$. We call the algebra A as solvable if $A^{(r)} = 0$ for some integer $r \geq 1$. The maximal solvable ideal of A is called the radical of A. When the radical of the algebra is the zero ideal, then the algebra is a semisimple algebra.*

Definition 3.7. *For an algebra A over a field F we obtain the descending central series of subalgebras as $A^1 \supseteq A^2 \supseteq A^3 \supseteq A^4 \dots$ by defining $A^1 = A$ and $A^{k+1} = A^k A$. The algebra A said to be nilpotent if $A^m = 0$ for some integer $m \geq 1$.*

Definition 3.8. *If A_1 and A_2 are two algebras, an algebra homomorphism is a linear map $f : A_1 \to A_2$ such that it preserves the multiplications of the corresponding algebras. We denote by $Hom(A_1, A_2)$, the collection of all algebra homomorphisms from A_1 to A_2.*

Definition 3.9. *If L is a Lie algebra, by a representation of L in a vector space V we mean a Lie algebra homomorphism $\rho : L \to End\ V$, where $End\ V$ is the Lie algebra consisting of all endomorphisms of V with the Lie multiplication $[g, h] = gh - hg$ for all g, h in $End\ V$.*

Definition 3.10. *Over the topological space X, a family of vector spaces consists of a topological space E along with the following:*

1. *a map q from E into X which is continuous*
2. *on each*
$$E_x = q^{-1}(x), \qquad for \quad x \in X$$
we have a finite dimensional vector space structure compatible with the topology on E_x and is induced from E.

Here, the map q is called the projection map, the topological space E is called the total space, the space X is called the base space and if $x \in X$ then E_x is called the fibre over x.

Definition 3.11. *A homomorphism from a family (E, p, X) to a family (F, q, X) of vector spaces is a continuous map $\varphi : E \to F$ such that:*

1. *$q\varphi = p$*
2. *for every x in X, the map $\varphi_x : E_x \to F_x$ is a linear map in each vector space.*

φ is said to be an isomorphism if φ is bijective and φ^{-1} is continuous. The families are isomorphic if an isomorphism exists between E and F.

Let V be a vector space and F be the space $X \times V$. If $q : F \to X$ is the projection of F onto its first factor then F is termed as the product family having fibre V. If any other family is isomorphic to a product family then it is called a trivial family.

Definition 3.12. *A family F of vector spaces over the space X is said to be a locally trivial family if for every x in the space X, there is a neighborhood U such that $F|_U$ is trivial. We denote the vector bundle by (F, p, X) and sometimes by F itself. Any subfamily of a vector bundle F is a subbundle of F if it is locally trivial.*

Definition 3.13. *For a vector bundle (E, p, X), a section of E is a continuous map $S : X \to E$ such that for all x in X, $p \circ S(x) = x$.*

Definition 3.14. *The collection of all sections of a vector bundle E forms a vector space and we denote it by $\Gamma(E)$.*

Definition 3.15. *Let (E, p, X) and (F, q, X) be two vector bundles. Consider the set $E \oplus F = \bigcup_{x \in X} (E_x \times F_x)$ and the map $p \oplus q : E \oplus F \to X$ given by $(p \oplus q)|_{E_x \oplus F_x} = \{x\}$. Then $E \oplus F$ is a vector bundle and is called the direct Whitney sum of E and F.*

We denote by $Mor(E, F)$, the $\mathcal{C}(X)-$module of all vector bundle morphisms from E to F. The vector bundle $Hom(E, F) = \bigcup_{x \in X} Hom(E_x, F_x)$, where $Hom(E_x, F_x)$ is the vector space of all linear maps from E_x to F_x. The vector bundle $Hom(E, E)$ is usually denoted by $End(E)$.

Definition 3.16. *A Lie algebra bundle is defined to be a vector bundle $\xi = (\xi, p, X)$ in which each of its fibre ξ_x is a Lie algebra. For each x in X, we can find an open neighbourhood V of x, a Lie algebra L and a homeomorphism $\phi : V \times L \to p^{-1}(V)$ such that for each point y in V, the map $\phi_y : L \to p^{-1}(y)$ is a Lie algebra isomorphism.*

A more formal definition of a Lie algebra bundle is given in [32].

Definition 3.17. *A morphism $\varphi : \xi \to \zeta$ of Lie algebra bundles ξ and ζ over the base space X is a continuous map. For each x in X, $\varphi_x : \xi_x \to \zeta_x$ is a Lie algebra homomorphism. We call a morphism φ as an isomorphism if φ is bijective and φ^{-1} is continuous.*

Definition 3.18. *A vector bundle $\xi = (\xi, p, X)$ with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces a Lie algebra structure on every fibre ξ_x is called a weak Lie algebra bundle.*

Definition 3.19. *A subalgebra bundle of Lie algebra bundle ξ is a vector subbundle whose fibres are subalgebras. In short, we refer to a subalgebra bundle as a subbundle. Furthermore, an ideal bundle is a subbundle in which each fibre is an ideal.*

Definition 3.20. *We call a Lie algebra bundle ξ to be semisimple if the fibres $\xi_x, x \in X$ are semisimple Lie algebras [50, 51].*

Definition 3.21. *A Lie algebra bundle ξ is said to be abelian if each of its fibre is abelian.*

Definition 3.22. *If ξ is a Lie algebra bundle, then the derived series of ξ is defined as: $\xi^{(0)} = \xi$, $\xi^{(1)} = [\xi, \xi] = \bigcup_{x \in X} [\xi_x, \xi_x]$ and $\xi^{(k)} = \bigcup_{x \in X} [\xi_x^{(k-1)}, \xi_x^{(k-1)}]$ for a non negative integer k.*

ξ is a solvable Lie algebra bundle if $\xi^{(m)} = 0$ for a positive integer m.

Definition 3.23. *If ξ is a Lie algebra bundle, the descending central series of ξ is defined as: $\xi^0 = \xi$, $\xi^1 = [\xi, \xi] = \bigcup_{x \in X} [\xi_x, \xi_x]$ and $\xi^k = \bigcup_{x \in X} [\xi_x^{k-1}, \xi_x]$ for any non negative integer k.*

ξ is a nilpotent Lie algebra bundle if each of its fibre is nilpotent.

Definition 3.24. *Let ξ be a Lie algebra bundle and $\mathfrak{R}_x$ be the radical of the Lie algebra ξ_x. Then $\mathfrak{R} = \bigcup_{x \in X} \mathfrak{R}_x$ becomes an ideal bundle of ξ and is termed as the radical bundle of ξ.*

Definition 3.25. *If ξ is a Lie algebra bundle and $Z(\xi_x)$ is the center of the Lie algebra ξ_x then $Z(\xi) = \bigcup_{x \in X} Z(\xi_x)$ becomes an ideal bundle of ξ and we call it the center of ξ.*

3.1.3 Notations and terminologies

The vector spaces considered in this thesis are real or complex, having finite dimensions. If (E, p, X) is a Lie algebra bundle, it is denoted as ξ and its fibres as ξ_x. Unless specified otherwise, all bundles are over the same base space X. Locally trivial Lie algebra bundles are termed as just Lie algebra bundles. The base space X is assumed to be compact Hausdorff in the chapters 2-6.

3.2 CHAPTER 2: ALGEBRAIC LIE ALGEBRA BUNDLES

A group G of matrices over complex numbers is said to be algebraic group if the criteria for a matrix A to belong to G can be expressed as a collection of algebraic conditions in coefficients of A [12]. In this chapter, algebraic Lie algebra bundles are defined and a few basic properties of replicas are discussed. It has been proved that the derivation algebra bundle of a Lie algebra bundle is l-algebraic. The chapter concludes with a necessary condition for the radical bundle of an algebraic subbundle to be l-algebraic. algebraic. The chapter concludes with a necessary condition for the radical bundle of an algebraic subbundle to be l-algebraic.

3.2.1 Algebraic Lie algebra bundles

Definition 3.26. *Let $\xi = (E, p, X)$ be a Lie algebra bundle with local trivialization $\phi : U \times L \to p^{-1}(U)$. For any endomorphism A of ξ, we define $A' \in End(\xi)$ as the* ***replica*** *of A, if any invariant of $A|_{\xi_x}$ is also an invariant of $A'|_{\xi_x}$. The set of all replicas of A is denoted by $\{A\}$.*

Definition 3.27. *Any subbundle of $End(\xi)$ is said to be an* ***l-algebraic bundle*** *if each of its fibre is $l - algebraic$.*

Lemma 3.28. *Let $\xi = (E, p, X)$ be a Lie algebra bundle. Then any $A \in End(\xi)$ can be uniquely expressed as $A = A^0 + A^s$ where A^0 is nilpotent and A^s on ξ_x is a matrix with simple elementary divisors.*

Lemma 3.29. *In the representation of Lemma 3.28 , A^0 and A^s are replicas of A.*

Lemma 3.30. *Any $A \in End(\xi)$ can be decomposed as $A_x = (A_0)_x + \lambda_1(A_1)_x + \lambda_2(A_2)_x + \ldots + \lambda_l(A_l)_x$ on each fibre ξ_x of $End(\xi)$.*

Theorem 3.31. *If ξ is a Lie algebra bundle then the derivation algebra bundle of ξ, $\mathcal{D}(\xi)$ is l-algebraic.*

Definition 3.32. *Let $\xi = (E, p, X)$ be a Lie algebra bundle and $End(\xi)$ be the Lie algebra bundle of endomorphisms on ξ with the local trivialization $\phi : U \times End(L) \to (Endp)^{-1}(U)$ such that $\phi_x : \{x\} \times End(L) \to (Endp)^{-1}(x)$ is an isomorphism. By this isomorphism, for a subbundle η of $End(\xi)$ there exists a smallest algebraic Lie algebra η_x^* containing η_x for all $x \in X$. Then $\eta^* = \bigcup_{x \in X} \eta_x^*$ is a Lie algebra bundle and is called the algebraic hull of η.*

From the definition, we observe that the algebraic hull η^* is the smallest ideal bundle of ξ containing η.

Proposition 3.33. *Let η be a Lie subbundle of $End(\xi)$. Then $I(\eta)^* = I(\eta^*)$.*

Remark 3.34. *For a Lie subalgebra L of $gl(V)$, $R(L^*) = R(L)^*$[30]. For a subbundle η of End (ξ), $\mathfrak{R}(\eta) = \bigcup_{x \in X} \mathfrak{R}(\eta_x)$ and hence $\mathfrak{R}(\eta^*) = \mathfrak{R}(\eta)^*$.*

Proposition 3.35. *Let η^* be the algebraic hull of a subbundle η of $End(\xi)$. Then*

1. *Every ideal bundle of η is also an ideal bundle in η^*.*

2. *Center of η is contained in the center of η^*.*

3. *η and η^* have the same derived algebra bundle.*

4. *if η is an ideal bundle in a subbundle η_1 of $End(\xi)$ then $[\eta_1^*, \eta^*] \subset \eta$.*

Corollary 3.36. *If η^* is the algebraic hull of a subbundle η of $End(\xi)$ then η is an ideal in η^* and η^*/η is an abelian subbbundle. If η is solvable, η^* is solvable.*

Theorem 3.37. *Let η be an algebraic Lie subbundle of $End(\xi)$. If $\mathfrak{R}(\eta)$ is the radical of η and $\eta = \mathfrak{S} + \mathfrak{R}(\eta)$ then $\mathfrak{R}(\eta)$ is algebraic.*

3.3 CHAPTER 3: DERIVATIONS OF LIE ALGEBRA BUNDLES

A derivation of a Lie algebra plays an important role in the structure of Lie algebra. We begin with the result that if a Lie algebra bundle is the direct sum of ideal bundles then its derivation algebra bundle can be written as sum of derivation algebra bundles between each ideal bundles. We define characteristically solvable Lie algebra bundles and prove a characterization for the same. The relations between inner and central derivations of a Lie algebra bundle are discussed. A necessary and sufficient condition for a derivation to be an inner derivation is obtained and relation between inner derivations of Lie algebra bundle and its radical ideal bundle is studied.

Let ξ be a Lie algebra bundle. Any vector bundle morphism $D : \xi \to \xi$ is called a **derivation** if $D[u, v] = [u, Dv] + [Du, v]$ for all $u, v \in \xi_x$. A derivation D is called **inner** if there exists a section S of ξ such that for all u in ξ_x and x in X, $Du = [u, S(x)]$[63]. We define central derivation of a Lie algebra bundle as follows.

Definition 3.38. *A derivation D is **central** if $D\xi \subseteq Z(\xi)$.*

The set of all derivations of ξ form a Lie algebra bundle is denoted by $\mathcal{D}(\xi)$. Then the set of all inner derivations of ξ denoted by $\mathcal{I}(\xi)$ and the set of all central derivations of ξ denoted by $\mathcal{C}(\xi)$ form subbundles of $\mathcal{D}(\xi)$.

Let $\xi = (E, p, X)$ be a Lie algebra bundle and ξ be the direct sum of the ideal bundles $\xi_1, \xi_2, \ldots, \xi_n$. Let p_i be the projection morphism of ξ onto ξ_i. Identify an element φ_{ij} of $End(\xi_i, \xi_j)$ with an element $\varphi_{ij}p_i$ of $End(\xi)$. Then we can say that $End(\xi_i, \xi_j) \subset End(\xi)$. Let $\mathcal{D}(\xi_i, \xi_j) = \mathcal{D}(\xi) \cap End(\xi_i, \xi_j)$. It follows that $\mathcal{D}(\xi_i, \xi_i) = \mathcal{D}(\xi_i)$.

Theorem 3.39. *Let ξ be the direct sum of the ideal bundles ξ_1, ξ_2,... ξ_n. Then*

1. *$\mathcal{D}(\xi) = \sum_{i,j=1}^{n} \mathcal{D}(\xi_i, \xi_j)$.*

2. *For $i \neq j, \mathcal{D}(\xi_i, \xi_j)$ consists of $\varphi_{ij} \in End(\xi_i, \xi_j)$ such that $\varphi_{ij}\xi_i \subset Z(\xi_j)$ and $\varphi_{ij}[\xi_i, \xi_i] = 0$.*

3. *For $i \neq j, \mathcal{D}(\xi_i, \xi_j)$ is abelian.*

Definition 3.40. *A Lie algebra bundle $\xi = (E, p, X)$ is said to be characteristically solvable if $\mathcal{D}(\xi)$ is solvable and $Z(\xi) \subset [\xi, \xi]$.*

Theorem 3.41. *Let ξ be a solvable Lie algebra bundle such that $Z(\xi) \subset [\xi, \xi]$. If $\mathcal{D}(\xi) = \mathfrak{S} + \mathfrak{R}(\xi)$ then ξ is characteristically solvable.*

Theorem 3.42. *Suppose $\xi = (E, p, X)$ be the direct sum of the ideal bundles $\xi_1, \xi_2, \ldots, \xi_n$. Suppose $Z(\xi_j) \subset [\xi_j, \xi_j]$ for some j, then*

1. *$\mathcal{C}(\xi_j)$ is an abelian ideal bundle of $\mathcal{D}(\xi)$;*
2. *$[\mathcal{D}(\xi_i, \xi_j), \mathcal{D}(\xi_j, \xi_i)] \subset \mathcal{C}(\xi_j)$ for all $i \neq j$.*

Theorem 3.43. *Let ξ be a nonabelian solvable Lie algebra bundle. If $\mathcal{D}(\xi) = \mathfrak{S} \oplus \mathfrak{R}$ then $\mathcal{D}(\xi)$ is solvable and ξ is either characteristically solvable or the direct sum of a characteristically solvable ideal bundle and a central ideal bundle of rank 1.*

From [6], $Herm(E) = \bigcup_{x \in X} Herm(E_x)$ forms a vector bundle where $Herm(E_x)$ is the vector space of all Hermitian forms on E_x.
For $x \in X$ and $\forall f, g \in Herm(E_x)$ define $[f, g] = f \circ g - g \circ f$. Then $Herm(E_x)$ becomes a Lie algebra with this product structure. For every $x \in X$ and a neighbourhood U of x in X let $\phi : p^{-1}(U) \to U \times \xi_x$ be the local trivialization of ξ.

Define, $Herm\phi : U \times Herm(E_x) \to (Hermp)^{-1}(U)$ by $Herm\phi(s, T) = \phi_s \circ T \circ (\phi_s)^{-1}$.

We observe that $Herm\phi$ is a Lie bundle isomorphism. Hence $Herm(E)$ is a Lie algebra bundle.

Definition 3.44. *A Hermitian metric on a Lie algebra bundle is a section $h : E \to HermE$ such that $h(x)$ is positive definite for all $x \in X$. A Lie bundle with a specified Hermitian metric is called a Hermitian Lie bundle.*

Proposition 3.45. *Let $\xi = (E, p, X)$ be a Lie algebra bundle, η be a subbundle of ξ and h be the Hermitian metric on ξ. Then there exists a subbundle η' of ξ such that $\xi = \eta \oplus \eta'$.*

Proposition 3.46. *Let ξ be any subbundle of the Lie algebra bundle $End\,(\eta)$. If $\mathcal{C}(\xi) \subset \mathcal{I}(\xi)^*$ then $Z(\xi) = 0$ or $\xi = [\xi, \xi]$.*

We shall study Lie algebra bundles whose derivations are inner and central at the same time.

Let $\xi = (E, p, X)$ be a Lie algebra bundle. For all $x \in X$, set $Z_1(\xi_x) = \{e \in \xi_x : [e, \xi_x] \subseteq Z(\xi_x)\}$. Then $Z_1(\xi_x)$ is a subalgebra of ξ_x, $x \in X$. Define $Z_1(\xi) = \bigcup_{x \in X} Z_1(\xi_x)$. Then $Z_1(\xi)$ is a subbundle of ξ.

Proposition 3.47. *Let $Z(\xi)$ be the center of ξ. Then $\mathcal{I}(\xi) \cap \mathcal{C}(\xi) = \mathcal{I}(Z_1(\xi))$.*

The following theorem is obtained if the inner derivations of a Lie algebra bundle are all central.

Theorem 3.48. *$\mathcal{I}(\xi) \subset \mathcal{C}(\xi)$ if and only if $\xi^3 = 0$.*

The following theorem gives insight into Lie algebra bundles whose inner and central derivations are the same.

Theorem 3.49. *Assume that the center $Z(\xi_x)$ is non zero for each $x \in X$. Then $\mathcal{I}(\xi) = \mathcal{C}(\xi)$ if and only if $\xi^2 = Z(\xi)$ and rank $Z(\xi) = 1$.*

A necessary and sufficient condition for all the derivations of a Lie algebra bundle to be central derivations is obtained below.

Theorem 3.50. *$\mathcal{D}(\xi) = \mathcal{C}(\xi)$ if and only if ξ is abelian.*

Theorem 3.51. *Let ξ be the direct sum of the ideal bundles $\xi_1, \xi_2, ..., \xi_n$. Then $\mathcal{D}(\xi) = \mathcal{I}(\xi) \oplus \mathcal{C}(\xi)$ if and only if $\mathcal{D}(\xi_i) = \mathcal{I}(\xi_i) \oplus \mathcal{C}(\xi_i)$.*

Theorem 3.52. *Let ξ be a non abelian nilpotent Lie algebra bundle such that $Z(\xi)$ is not contained in $[\xi, \xi]$. Then $\mathcal{D}(\xi)$ is not nilpotent. $\mathcal{D}(\xi)$ contains a solvable non nilpotent ideal bundle.*

3.3.1 Derivation algebra bundle and radical bundle

In this section we observe that the radical bundle of any Lie algebra bundle and its derivation algebra bundle are closely related. We define splitting of Lie algebra bundles and study the relations between inner derivations of a Lie algebra bundle and its radical bundle.

Definition 3.53. *Let ξ be a Lie algebra bundle and ξ_1 be a subbundle of ξ. If there exists a subbundle ξ_2 of ξ such that $\xi = \xi_1 + \xi_2$ and $\xi_1 \cap \xi_2$ is the trivial bundle then we say that ξ splits over ξ_1 and ξ_2 is called a complement of ξ_1 in ξ.*

Lemma 3.54. *A semisimple ideal bundle of ξ and its complementary ideal bundle in ξ are both characteristic.*

Remark 3.55. *For a Lie algebra L, if I and J are semisimple ideals of L then, $I + J$ is a semisimple ideal of L. If $\phi : U \times L \to \bigcup_{x \in U} \xi_x$ is the local triviality of ξ and $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are semisimple ideal bundles of ξ then using the the above lemma, $\phi|_{(I+J)} : U \times (I + J) \to \bigcup_{x \in U} ((\mathfrak{S}_1)_x + (\mathfrak{S}_2)_x)$ is the local triviality of the semisimple ideal bundle $\mathfrak{S}_1 + \mathfrak{S}_2$. Thus we can obtain the sum of all the semisimple ideal bundles of ξ which is the largest semisimple ideal bundle of ξ.*

Lemma 3.56. *Let $\mathfrak{R}$ and $\mathfrak{N}$ be the radical bundle and the largest nilpotent ideal bundle of ξ. If $\mathcal{S}$ is a maximal semisimple subbundle of the complementary ideal bundle of the largest semisimple ideal bundle of ξ then no sections of $\mathcal{S}$ induce inner derivations of $\mathfrak{R}$ and $\mathfrak{N}$.*

Lemma 3.57. *$\mathcal{I}(\xi)|_{\mathfrak{R}}$ and $\mathcal{I}(\mathfrak{R})$ coincide if and only if ξ is the direct sum of a semisimple ideal bundle and the radical bundle.*

Lemma 3.58. *$\mathcal{I}(\xi)|_{\mathfrak{R}}$ splits over $\mathcal{I}(\mathfrak{R})$.*

With the help of the above lemmas we prove the following theorem which characterizes the inner derivations of a Lie algebra bundle.

Theorem 3.59. *Let ξ be a Lie algebra bundle and $\mathfrak{R}$ be its radical bundle. Then any derivation D of ξ is inner if and only if there exists a section S of ξ such that $D|_{\mathfrak{R}_x} = [S(x), \mathfrak{R}_x]$ for all x in X.*

Theorem 3.60. *Let ξ be a Lie algebra bundle and $\mathfrak{R}$ be the radical bundle of ξ. Then $\mathcal{D}(\xi) = \mathcal{I}(\xi)$ if and only if $\mathcal{D}(\xi)|_{\mathfrak{R}} = \mathcal{I}(\xi)|_{\mathfrak{R}}$.*

Corollary 3.61. *If every derivation of the radical bundle $\mathfrak{R}$ can be extended to an inner derivation of ξ then all the derivations of ξ are inner.*

Corollary 3.62. *Let ξ be a Lie algebra bundle and $\mathfrak{R}$ be its radical bundle. Then any derivation of $\mathfrak{R}$ is inner if and only if*

1. *Every derivation of ξ induces an inner derivation of the radical bundle $\mathfrak{R}$.*
2. *Every derivation of ξ is inner and ξ is the sum of a semisimple ideal bundle and $\mathfrak{R}$.*

Following Theorem 3.59 and Lemma 3.58 we have,

Theorem 3.63. *Let $\mathfrak{R}$ be the radical ideal bundle of the Lie algebra bundle ξ. Then the following conditions are equivalent.*

1. *$\mathcal{D}(\xi)$ splits over $\mathcal{I}(\xi)$.*
2. *$\mathcal{D}(\xi)|_{\mathfrak{R}}$ splits over $\mathcal{I}(\xi)|_{\mathfrak{R}}$.*
3. *$\mathcal{D}(\xi)|_{\mathfrak{R}}$ splits over $\mathcal{I}(\mathfrak{R})$.*

Corollary 3.64. *If every derivation of $\mathfrak{R}$ can be extended to a derivation of ξ then $\mathcal{D}(\xi)$ splits over $\mathcal{I}(\xi)$ if and only if $\mathcal{D}(\mathfrak{R})$ splits over $\mathcal{I}(\mathfrak{R})$.*

3.4 CHAPTER 4: PRIME, SEMIPRIME AND IRREDUCIBLE IDEAL BUNDLES

Prime ideals play an important role in the theory of associative algebras.It is of interest how the notion of prime ideals can be applied to Lie algebra bundles. In this chapter we define prime ideal bundle of a Lie algebra bundle and characterize it in terms of sections of Lie algebra bundle. Interrelations between prime, semi prime and irreducible Lie algebra bundles are deduced.

3.4.1 Prime, semiprime and irreducible ideal bundles

Definition 3.65. *An ideal bundle P of a Lie algebra bundle ξ is said to be a prime ideal bundle if for each x in X, P_x is a prime ideal of ξ_x.*

Theorem 3.66. *Let P be an ideal bundle of a Lie algebra bundle ξ. Then the following conditions are equivalent.*

1. *P is a prime ideal bundle.*
2. *If $[S(x), H_x] \subseteq P_x$ for some section S of ξ and ideal bundle H of ξ then either $S(x) \in P_x$ or $H_x \subseteq P_x$.*
3. *If $[S_1(x), \langle S_2(x)\rangle] \subseteq P_x$ for sections S_1, S_2 of ξ then $S_1(x) \in P_x$ or $S_2(x) \in P_x$.*

Definition 3.67. *An ideal subbundle Q of a Lie algebra bundle ξ is said to be* ***semiprime*** *if for ideal bundle H of ξ, $[H, H] \subseteq Q$ then $H \subseteq Q$.*

Definition 3.68. *An ideal bundle N of a Lie algebra bundle ξ is said to be* ***irreducible*** *if $N = H \cap K$ for ideal bundles H and K of ξ gives $N = H$ or $N = K$.*

Lemma 3.69. *Let ξ be a Lie algebra bundle. Then*

1. *Any prime ideal bundle is a semiprime ideal bundle.*
2. *Any prime ideal bundle is irreducible.*

Definition 3.70. *A Lie algebra bundle ξ is said to satisfy the minimal condition for ideal bundles if for every chain of ideal bundles $\eta_1 \supseteq \eta_2 \supseteq \dots$ there exists an index m such that $\eta_i = \eta_k$ for every i, $k > m$.*

We write $\xi \in Min - \lhd$ when ξ satisfies the minimal condition for ideal bundles. Similarly, we have

Definition 3.71. *A Lie algebra bundle ξ is said to satisfy the maximal condition for ideal bundles if for every chain of ideal bundles $\eta_1 \subseteq \eta_2 \subseteq \dots$ there exists an index m such that $\eta_i = \eta_k$ for every i, $k > m$.*

We write $\xi \in Max - \lhd$ when ξ satisfies the maximal condition for ideal bundles.

Proposition 3.72. *Let ξ be a Lie algebra bundle and P be a prime ideal bundle of ξ.*

1. *P is prime if and only if P is irreducible and semiprime.*
2. *Let $\xi \in Min - \lhd$. Then an ideal bundle P of ξ is prime if and only if there exists the smallest ideal bundle M which contains P, $M \neq P$ such that M/P is not abelian.*

Definition 3.73. *An ideal bundle M of Lie algebra bundle ξ, $M \neq \xi$, is said to be maximal if for each x in X, M_x is a maximal ideal of ξ_x.*

Then, if M' is any maximal ideal bundle of ξ and $M \subseteq M' \subseteq \xi$ we have $M = M'$ or $M' = \xi$.

Theorem 3.74. *Let M be a maximal ideal bundle of a Lie algebra bundle ξ. Then the following conditions are equivalent.*

1. *M is a prime ideal bundle.*
2. *M is a semiprime ideal bundle.*
3. *$rank(\xi/M) > 1$.*

Definition 3.75. *A Lie algebra bundle ξ is said to be perfect if $\xi = [\xi, \xi]$.*

Corollary 3.76. *Let ξ be a perfect Lie algebra bundle. Then a maximal ideal bundle of ξ is prime.*

Corollary 3.77. *Let ξ be a semisimple Lie algebra bundle and H be an ideal subbundle of ξ, $H \neq \xi$. Then the following are equivalent.*

1. *H is a prime ideal bundle.*

2. *H is a maximal ideal bundle.*

3. *H is an irreducible ideal bundle.*

Let ξ be a Lie algebra bundle with local triviality given by $\phi : U \times L \to p^{-1}(U)$ where L is a Lie algebra. η be an ideal subbundle of ξ with local triviality $\phi|_I : U \times I \to p^{-1}(U)$, I is an ideal of L.

Denote by $r(I)$, the intersection of prime ideals P of L containing I and $r(\eta_x)$, the intersection of prime ideals of ξ_x containing the ideal fibre η_x. Since $ker\phi_x = 0$, P contains $ker\,\phi_x$ and ϕ_x being an isomorphism, $\phi_x(P)$ is a prime ideal of ξ_x. Now, $I \subseteq P$ implies $\eta_x = \phi_x(I) \subseteq \phi_x(P)$ so that $\phi_x(P)$ is a prime ideal of ξ_x containing η_x. We have, $\phi_x(\cap_{i\in I} P) = \bigcap_{i\in I} \phi_x(P) \subseteq r(\eta_x)$ which gives $\phi_x(r(I)) \subseteq r(\eta_x)$. Similarly, $\phi_x^{-1}(r(\eta_x)) \subseteq r(I)$. Then $\phi \restriction_{U\times r(I)} : U \times r(I) \to \bigcup\limits_{x\in U} r(\eta_x)$ is an isomorphism and hence, $r(\eta) = \bigcup\limits_{x\in X} r(\eta_x)$ is an ideal subbundle of ξ.

In particular, when η is the zero ideal bundle, η_x for $x \in X$ is the intersection of all prime ideals of ξ_x. Then $r(0)$ is the union of all prime radicals of the fibres of ξ and we call $r(0)$ as the prime radical of ξ and denote it as $r(\xi)$.

Theorem 3.78. *$\Re(\xi)$ is contained in $r(\xi)$. If $\xi \in Max - \lhd$ then $\Re(\xi) = r(\xi)$.*

Theorem 3.79. *Any finite intersection of ideal bundles of a semisimple Lie algebra bundle ξ is an ideal bundle of ξ.*

Theorem 3.80. *Let ξ be a semisimple Lie algebra bundle. Then the finite intersection of prime ideal bundles of ξ is a semiprime ideal bundle of ξ.*

Corollary 3.81. *The prime radical bundle of any ideal bundle of ξ is a semiprime ideal bundle of ξ.*

Corollary 3.82. *Let η be an ideal bundle of ξ. If $A^2 \subseteq \eta$, A an ideal bundle of ξ, then $A \subseteq r(\eta)$.*

Lemma 3.83. *Let η be an ideal bundle of a Lie algebra bundle ξ. Then $r(\xi/\eta) = r(\eta)/\eta$.*

Theorem 3.84. *Let $\xi \in Max - \lhd$ and η be an ideal bundle of ξ. Then there exists a $n \in \mathbb{N}$ such that $(r(\eta))^{(n)} \subseteq \eta$.*

Theorem 3.85. *Let $\xi \in Max - \lhd$ and η be an ideal bundle of ξ. Then η has a finite number of minimal prime ideal bundles containing it.*

The following theorem gives characterization of semiprime ideal bundles when $\xi \in Max - \lhd$.

Theorem 3.86. *Let $\xi \in Max - \lhd$ and Q be an ideal bundle of ξ. Then the following statements are equivalent.*

1. *Q is a semiprime ideal bundle.*

2. *$Q = r(Q)$.*

3. *Q is a finite intersection of prime ideal bundles of ξ.*

Lemma 3.87. *Every ideal bundle of a semisimple Lie algebra bundle ξ is a semiprime ideal bundle.*

Definition 3.88. *An ideal bundle η of ξ which is different from ξ is said to be* ***strongly irreducible*** *if for any pair of ideal bundles ζ_1 and ζ_2 of ξ, $\zeta_1 \cap \zeta_2 \subseteq \eta$ implies $\zeta_1 \subseteq \eta$ or $\zeta_2 \subseteq \eta$.*

Theorem 3.89. *Any prime ideal bundle of a Lie algebra bundle ξ is strongly irreducible.*

Theorem 3.90. *Every strongly irreducible ideal bundle of a semisimple Lie algebra bundle ξ is prime.*

3.5 CHAPTER 5: PRIMARY IDEAL BUNDLES

In this chapter, we define primary ideal bundles of a Lie algebra bundle and study the relations between prime and primary ideal bundles. A characterisation of primary ideal bundles in terms of sections of Lie algebra bundles is done. We have deduced that if $\xi \in Max - \lhd$, then the notions of prime ideal bundles, primary ideal bundles and irreducible ideal bundles are equivalent. At the end we introduce the concept of prime Lie algebra bundles and primary Lie algebra bundles.

3.5.1 Primary ideal bundles

Definition 3.91. *An ideal bundle P of a Lie algebra bundle ξ is said to be a primary ideal bundle of ξ if the ideal η_x is a primary ideal in ξ_x.*

Theorem 3.92. *If $r(P)$ is a prime ideal bundle of ξ, then P is a primary ideal bundle of ξ.*

Corollary 3.93. *Every prime ideal bundle of ξ is primary.*

Definition 3.94. *A prime ideal bundle of ξ is called a* ***prime radical ideal bundle*** *of ξ if its prime radical bundle is itself that is, $r(P) = P$.*

Theorem 3.95. *Let P be a primary ideal bundle of ξ. If P is a prime radical ideal bundle of ξ then P is a prime ideal bundle of ξ.*

Lemma 3.96. *If M is an ideal bundle of ξ with $r(M) = \xi$, then M is a primary ideal bundle.*

Lemma 3.97. *If M is a maximal prime radical ideal bundle of ξ then M is a prime ideal bundle of ξ.*

Theorem 3.98. *Any maximal ideal bundle M of ξ is a primary ideal bundle of ξ.*

Theorem 3.99. *Let P be an ideal bundle of a Lie algebra bundle ξ. Then the following conditions are equivalent.*

1. *P is a primary ideal bundle.*

2. *If* $[S(x), H_x] \subseteq P_x$ *for some section* S *of* ξ *and ideal bundle* H *in* ξ *then either* $S(x) \in r(P_x)$ *or* $H_x \subseteq r(P_x)$.

3. *If* $[S_1(x), \langle S_2(x)\rangle] \subseteq P_x$ *for some sections* S_1 *and* S_2 *of* ξ *then* $S_1(x) \in r(P_x)$ *or* $S_2(x) \in r(P_x)$.

Theorem 3.100. *Let* $\xi \in Max - \lhd$. *An ideal bundle* M *is semiprime and primary if and only if* M *is a prime ideal bundle.*

Theorem 3.101. *Let* $\xi \in Max - \lhd$. M *is semiprime and strongly irreducible if and only if* M *is a prime ideal bundle.*

Corollary 3.102. *Let* $\xi \in Max - \lhd$ *and* M *be a semiprime ideal bundle of* ξ. *Then the following conditions are equivalent:*

1. M *is a prime ideal bundle.*

2. M *is a strongly irreducible ideal bundle in* ξ.

3. M *is a primary ideal bundle.*

3.5.2 Prime and primary Lie algebra bundle

In this section, we define and discuss prime Lie algebra bundles and primary Lie algebra bundles.

Definition 3.103. *A Lie algebra bundle* ξ *is called a* ***prime Lie algebra bundle*** *if for nonzero ideal bundles* A, B *of* ξ, $[A, B] \neq 0$.

Definition 3.104. *A Lie algebra bundle* ξ *is called a* ***semiprime Lie algebra bundle*** *if for nonzero ideal bundle* A *of* ξ, $A^2 \neq 0$.

We observe that an ideal bundle of ξ is prime(semiprime) if and only if the zero bundle is a prime(semiprime) ideal bundle of ξ.

Theorem 3.105. *A Lie algebra bundle* ξ *having a non-zero center cannot be a semiprime Lie algebra bundle.*

Theorem 3.106. *A semisimple Lie algebra bundle is a semiprime Lie algebra bundle.*

Theorem 3.107. *For a proper ideal bundle* P *of Lie algebra bundle* ξ *the following are equivalent:*

1. P *is a prime ideal bundle.*

2. *If* A *and* B *are ideal bundles of* ξ *containing* P *properly then* $[A, B] \not\subseteq P$.

3. ξ/P *is a prime Lie algebra bundle.*

A similar result for primary Lie algebra bundles is as follows.

Theorem 3.108. *For a proper ideal bundle* P *of Lie algebra bundle* ξ *the following are equivalent:*

1. P *is a primary ideal bundle.*

2. *If* A *and* B *are ideal bundles of* ξ *containing* $r(P)$ *properly then* $[A, B] \not\subseteq P$.

3. ξ/P *is a primary Lie algebra bundle.*

3.6 CHAPTER 6: STRONG PRIMENESS IN LIE ALGEBRA BUNDLES

In Jordan systems there are a few characterizations of strong primeness [4]. An elemental characterization in the case of Lie algebras is given by Esther García and Miguel Gómez Lozano in [25]. It is of interest how the notion of strong primeness can be transferred to Lie algebra bundles. In this chapter, we define strong primeness in Lie algebra bundles and deduce an elemental characterization of strong primeness.

3.6.1 Strong primeness in Lie algebras

Definition 3.109. *An element u of ξ is called an* ***absolute zero divisor*** *of ξ if there exists a section S of ξ such that $u = S(x)$ for some x in X and $ad^2_{S(x)} = 0$.*

Definition 3.110. *A Lie algebra bundle ξ is said to be* ***nondegenerate*** *if for every section S of ξ and for every x in X, $S(x)$ is not an absolute zero divisor in ξ_x.*

Definition 3.111. *ξ is a* ***strongly prime*** *bundle if it is both prime and nondegenerate.*

Proposition 3.112. *Let ξ be a Lie algebra bundle and S, $T\in\ \Gamma(\xi)$ such that $[S,[T,\xi]] = 0$ and $[S,T] = 0$. Then $[T,[S,\xi]] = 0$ and $ad_{S(x)}ad_a ad_{T(x)} = -ad_{T(x)}ad_a ad_{S(x)}$ for $a \in \xi_x$. Moreover if ξ is nondegenerate, $[S,[T,\xi]] = 0$ implies $[S,T] = 0$.*

Lemma 3.113. *Let ξ be a nondegenerate Lie algebra bundle and $S,T \in \Gamma(\xi)$ such that $ad_{S(x)}ad_{T(x)} = 0$ for all $x \in X$. Then the product $ad_{z_1}ad_{z_2}\dots ad_{z_n}$ is zero in ξ_x if the number of $z_i = S(x)$, called as r and the number of $z_j = T(x)$, called as s satisfy the condition $s \neq 0$, $r \neq 0$ and $n+1 < 2(r+s)$.*

In the following discussions, for $x \in X$, $a \in \xi_x$ and $S,T \in \Gamma(\xi)$ we denote $ad_{S(x)}$ as X_S, $ad_{T(x)}$ as X_T and ad_a as A.

Proposition 3.114. *Let ξ be a nondegenerate Lie algebra bundle and $n \in \mathbb{N}$. For sections S,T of ξ and $x \in X$, let $X_S A_1 A_2 \dots A_n X_T = 0$ for $a_1, a_2, \dots, a_n \in \xi_x$. Then $X_S A_1 A_2 \dots A_k X_T = 0$ and $X_T A_1 A_2 \dots A_k X_S = 0$ for $a_1, a_2, \dots, a_k \in \xi_x$, $0 \leq k \leq n$. Moreover, $[S,T] = 0$.*

Proposition 3.115. *Let ξ be a nondegenerate Lie algebra bundle and $n \in \mathbb{N}$. Let S,T be nonzero sections of ξ such that when S and T are nonzero, $X_S A_1 A_2 \dots A_n X_T = 0$ for $x \in X$ and $a_1, a_2, \dots, a_n \in \xi_x$. Then there exists sections S',T' of ξ which are non zero where S, T are nonzero such that $X_{S'} A_1 A_2 \dots A_{n+1} X_{T'} = 0$ for every $a_1, a_2, \dots, a_{n+1} \in \xi_x$.*

Proposition 3.116. *Let ξ be a nondegenerate Lie algebra bundle and let S and T be sections of ξ such that $ad_{S(x)}\, ad_a\, ad_b\, ad_{T(x)} = 0$ for every $a,b \in \xi_x$. Then $T(x) \in Ann_{\xi_x}(Id_{\xi_x}(S(x)))$ and $S(x) \in Ann_{\xi_x}(Id_{\xi_x}(T(x)))$ where $(Id_{\xi_x}(S(x)))$ is the ideal of ξ_x generated by $S(x)$.*

Theorem 3.117. *A Lie algebra bundle ξ is strongly prime if and only if for every S, $T \in \Gamma(\xi)$ such that $[S,[T,\xi]] = 0$ we have $S = 0$ or $T = 0$.*

3.7 CHAPTER 7: CONCLUSION AND SCOPE FOR FUTURE WORK

In the course of our study, it became evident that many results concerning Lie algebras and vector bundles may be extended to Lie algebra bundles by altering the concepts and prescribing constraints on the underlying space appropriately. Because the notion of Lie algebras and vector bundles is prevalent in most fields of science, it provides the path for future development of the theory of Lie algebra bundles. We have included a handful of them below.

1. To study algebraicity of Lie algebra bundles different from $End\ \xi$.
2. To further study derivations of Lie algebra bundles.
3. To extend the concept of Root systems in Lie algebras to Lie algebra bundles and to classify them.
4. To study sections of Lie algebra bundles.
5. To study Hochschild cohomology modules of Lie algebra bundles.
6. To develop Lie algebra bundles in terms of characteristic ideal bundles over a field.
7. To study Module bundles over Lie algebra bundles.

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4 List of Publications/Papers Presented/ Communicated

4.1 Publications in Journals/ Communicated

1. Algebraic Lie Algebra Bundles and Derivations of Lie Algebra Bundles.
   M. V. Monica and R. Rajendra.
   International Electronic Journal of Algebra, 35(2024), 95-107.

2. On Prime ideal bundles of Lie Algebra Bundles.
   M. V. Monica and R. Rajendra
   Journal of Algebra and Related Topics (Accepted).

3. Primary ideal bundles of Lie Algebra Bundles.
   M. V. Monica and R. Rajendra
   Boletim da Sociedade Paranaense de Matemática (Accepted).

4.2 Papers presented in an International/National conference

1. Presented a Paper titled "Derivations of Lie Algebra Bundles" in the One Day International Conference on Recent Advances in Interdisciplinary Mathematical Sciences held at Government Science College, Bengaluru on 6th September, 2019.

2. Presented a Paper titled "Algebraic Lie Algebra Bundles" in the International Conference on Emerging Trends in Applied Mathematics, held at Mount Carmel College, Bengaluru during 21-22 April, 2022.

3. Presented a Paper titled "Prime Ideal Bundles of Lie Algebra Bundles" in the one day Online National Conference on Recent Research Trends in Mathematics, held at VTM NSS College, Dhanuvachapuram held on 25th March, 2023.

4.3 Workshops/Conference attended

1. Participated in the "School on K-Theory and its Applications" organized by the Indian Statistical Institute, Bengaluru during December 26, 2016 - January 7, 2017.

2. Attended International Conference on "Global Advancement of Mathematics(GAM-2019)", held at Acharya Institute of Graduate Studies, Bengaluru during 25-26 June, 2019.

3. Attended National Workshop on "Technical Document Preparation using LaTex" organized by the Dr. S. & S. S. Ghandhy Govt. Engineering College, Surat held during 08- 13th June 2020.

4. Attended Faculty Development Programme on "Python 3.4.3" by Reva University-Spoken Tutorial Project, IIT Bombay during 01 -20 January 2021.

Signature of the Candidate
(Monica M. V.)

Signature of the Guide
(Dr. Rajendra R)